Derivation of transport equation for turbulent kinetic energy, k

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23rd October 2016

1 Introduction

The transport equation of TKE (turbulent kinetic energy or k) describes how mean flow feeds kinetic energy into turbulence. The transport equation of TKE also plays a vital role in the development of turbulence models.

The instantaneous kinetic energy $k(t)$ is the sum of mean kinetic $\overline{K} = (\overline{U}_x^2 + \overline{U}_y^2 + \overline{U}_z^2)/2$ energy and turbulent kinetic energy $k = (\overline{u'}_x^2 + \overline{u'}_y^2 + \overline{u'}_z^2)/2$.

$$k(t) = \overline{K} + k$$

2 Derivation:

Assuming that the fluid is incompressible, we use Navier-Stokes equations.

Mass conservation equation:

$$\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = 0 \tag{1}$$

Momentum conservations equations in scalar form:

$$\frac{\partial U_x}{\partial t} + \left(U_x \frac{\partial U_x}{\partial x} + U_y \frac{\partial U_x}{\partial y} + U_z \frac{\partial U_x}{\partial z} \right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right) + S_{M_x} \tag{2}$$

$$\frac{\partial U_y}{\partial t} + \left(U_x \frac{\partial U_y}{\partial x} + U_y \frac{\partial U_y}{\partial y} + U_z \frac{\partial U_y}{\partial z} \right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 U_y}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} + \frac{\partial^2 U_y}{\partial z^2} \right) + S_{M_y} \tag{3}$$

$$\frac{\partial U_z}{\partial t} + \left(U_x \frac{\partial U_z}{\partial x} + U_y \frac{\partial U_z}{\partial y} + U_z \frac{\partial U_z}{\partial z} \right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 U_z}{\partial x^2} + \frac{\partial^2 U_z}{\partial y^2} + \frac{\partial^2 U_z}{\partial z^2} \right) + S_{M_z} \tag{4}$$

where, S_{M_x}, S_{M_y}, S_{M_z}, in equations (2), (3), (4) are source terms and may contain body forces such as gravitation force, centrifugal force, Coriolis force, electromagnetic force, etc.

The instantaneous variables in the equations (2), (3) and (4) can be replaced with sum of their mean and fluctuating components e.g. $\overrightarrow{U} = \overline{U} + u' = (\overline{U_x} + u'_x)i + (\overline{U_y} + u'_y)j + (\overline{U_z} + u'_z)k$, $p = \overline{p} + p'$. Therefore, replacing the variables in equation (2) and neglecting the source terms in the present case, we obtain:

$$
\begin{aligned}
\frac{\partial(\overline{U_x}+u'_x)}{\partial t}+(\overline{U_x}+u'_x)\frac{\partial(\overline{U_x}+u'_x)}{\partial x}+(\overline{U_y}+u'_y)\frac{\partial(\overline{U_x}+u'_x)}{\partial y}\\
+(\overline{U_z}+u'_z)\frac{\partial(\overline{U_x}+u'_x)}{\partial z}=-\frac{1}{\rho}\frac{\partial(\overline{p}+p')}{\partial x}\\
+\nu\left(\frac{\partial^2(\overline{U_x}+u'_x)}{\partial x^2}+\frac{\partial^2(\overline{U_x}+u'_x)}{\partial y^2}+\frac{\partial^2(\overline{U_x}+u'_x)}{\partial z^2}\right)
\end{aligned}
\tag{5}
$$

Further multiplying equation (5) with u'_x, yields:

$$
\begin{aligned}
u'_x\frac{\partial}{\partial t}(\overline{U_x}+u'_x)+u'_x(\overline{U_x}+u'_x)\frac{\partial(\overline{U_x}+u'_x)}{\partial x}+u'_x(\overline{U_y}+u'_y)\frac{\partial(\overline{U_x}+u'_x)}{\partial y}\\
u'_x(\overline{U_z}+u'_z)\frac{\partial(\overline{U_x}+u'_x)}{\partial z}=-\frac{u'_x}{\rho}\frac{\partial}{\partial x}(\overline{p}+p')\\
+\nu u'_x\left[\frac{\partial^2(\overline{U_x}+u'_x)}{\partial x^2}+\frac{\partial^2(\overline{U_x}+u'_x)}{\partial y^2}+\frac{\partial^2(\overline{U_x}+u'_x)}{\partial z^2}\right]
\end{aligned}
\tag{6}
$$

The equation (6) can be further arranged to:

$$
\begin{aligned}
u'_x\frac{\partial\overline{U_x}}{\partial t}+u'_x\frac{\partial u'_x}{\partial t}+\overline{U_x}u'_x\frac{\partial\overline{U_x}}{\partial x}+\overline{U_x}u'_x\frac{\partial u'_x}{\partial x}+u'^2_x\frac{\partial\overline{U_x}}{\partial x}+u'^2_x\frac{\partial u'_x}{\partial x}+\overline{U_y}u'_x\frac{\partial\overline{U_x}}{\partial y}+\overline{U_y}u'_x\frac{\partial u'_x}{\partial y}\\
+u'_xu'_y\frac{\partial\overline{U_x}}{\partial y}+u'_xu'_y\frac{\partial u'_x}{\partial y}+\overline{U_z}u'_x\frac{\partial\overline{U_x}}{\partial z}+\overline{U_z}u'_x\frac{\partial u'_x}{\partial z}+u'_xu'_z\frac{\partial\overline{U_x}}{\partial z}+u'_xu'_z\frac{\partial u'_x}{\partial z}\\
=\frac{1}{\rho}u'_x\frac{\partial\overline{p}}{\partial x}+\frac{1}{\rho}u'_x\frac{\partial p'}{\partial x}+\nu u'_x\left[\frac{\partial^2\overline{U_x}}{\partial x^2}+\frac{\partial^2\overline{U_y}}{\partial y^2}+\frac{\partial^2\overline{U_z}}{\partial z^2}\right]+\nu u'_x\left[\frac{\partial^2u'_x}{\partial x^2}+\frac{\partial^2u'_y}{\partial y^2}+\frac{\partial^2u'_z}{\partial z^2}\right]
\end{aligned}
\tag{7}
$$

Taking the time average of equation (7):

$$
\begin{aligned}
\overline{u'_x\frac{\partial\overline{U_x}}{\partial t}}+\overline{u'_x\frac{\partial u'_x}{\partial t}}+\overline{\overline{U_x}u'_x\frac{\partial\overline{U_x}}{\partial x}}+\overline{\overline{U_x}u'_x\frac{\partial u'_x}{\partial x}}+\overline{u'^2_x\frac{\partial\overline{U_x}}{\partial x}}+\overline{u'^2_x\frac{\partial u'_x}{\partial x}}+\overline{\overline{U_y}u'_x\frac{\partial\overline{U_x}}{\partial y}}+\overline{\overline{U_y}u'_x\frac{\partial u'_x}{\partial y}}\\
+\overline{u'_xu'_y\frac{\partial\overline{U_x}}{\partial y}}+\overline{u'_xu'_y\frac{\partial u'_x}{\partial y}}+\overline{\overline{U_z}u'_x\frac{\partial\overline{U_x}}{\partial z}}+\overline{\overline{U_z}u'_x\frac{\partial u'_x}{\partial z}}+\overline{u'_xu'_z\frac{\partial\overline{U_x}}{\partial z}}+\overline{u'_xu'_z\frac{\partial u'_x}{\partial z}}\\
=\overline{\frac{1}{\rho}u'_x\frac{\partial\overline{p}}{\partial x}}+\overline{\frac{1}{\rho}u'_x\frac{\partial p'}{\partial x}}+\overline{\nu u'_x\left[\frac{\partial^2\overline{U_x}}{\partial x^2}+\frac{\partial^2\overline{U_y}}{\partial y^2}+\frac{\partial^2\overline{U_z}}{\partial z^2}\right]}+\overline{\nu u'_x\left[\frac{\partial^2u'_x}{\partial x^2}+\frac{\partial^2u'_y}{\partial y^2}+\frac{\partial^2u'_z}{\partial z^2}\right]}
\end{aligned}
\tag{8}
$$

Applying Reynolds averaging to the above equation (8)

$$
\begin{aligned}
\overline{\overline{\phi}\psi}&=\overline{\phi}\;\overline{\psi}\\
\overline{\overline{\phi}\phi'}&=\overline{\phi\phi'}=0\\
\overline{\phi+\psi}&=\overline{\phi}+\overline{\psi}
\end{aligned}
\tag{9}
$$

We obtain:

$$
\begin{aligned}
\underbrace{\overline{u'_x\frac{\partial\overline{U_x}}{\partial t}}}_{=0}+\overline{u'_x\frac{\partial u'_x}{\partial t}}+\underbrace{\overline{\overline{U_x}u'_x\frac{\partial\overline{U_x}}{\partial x}}}_{=0}+\overline{\overline{U_x}u'_x\frac{\partial u'_x}{\partial x}}+\overline{u'^2_x\frac{\partial\overline{U_x}}{\partial x}}+\overline{u'^2_x\frac{\partial u'_x}{\partial x}}+\underbrace{\overline{\overline{U_y}u'_x\frac{\partial\overline{U_x}}{\partial y}}}_{=0}+\overline{\overline{U_y}u'_x\frac{\partial u'_x}{\partial y}}\\
+\overline{u'_xu'_y\frac{\partial\overline{U_x}}{\partial y}}+\overline{u'_xu'_y\frac{\partial u'_x}{\partial y}}+\underbrace{\overline{\overline{U_z}u'_x\frac{\partial\overline{U_x}}{\partial z}}}_{=0}+\overline{\overline{U_z}u'_x\frac{\partial u'_x}{\partial z}}+\overline{u'_xu'_z\frac{\partial\overline{U_x}}{\partial z}}+\overline{u'_xu'_z\frac{\partial u'_x}{\partial z}}\\
=\underbrace{\overline{\frac{1}{\rho}u'_x\frac{\partial\overline{p}}{\partial x}}}_{=0}+\overline{\frac{1}{\rho}u'_x\frac{\partial p'}{\partial x}}+\underbrace{\overline{\nu u'_x\left[\frac{\partial^2\overline{U_x}}{\partial x^2}+\frac{\partial^2\overline{U_y}}{\partial y^2}+\frac{\partial^2\overline{U_z}}{\partial z^2}\right]}}_{=0}+\overline{\nu u'_x\left[\frac{\partial^2u'_x}{\partial x^2}+\frac{\partial^2u'_y}{\partial y^2}+\frac{\partial^2u'_z}{\partial z^2}\right]}
\end{aligned}
\tag{10}
$$

The equation (10) is further simplified to:

$$
\begin{aligned}
&\overline{u'_x\frac{\partial u'_x}{\partial t}} + \overline{U_x u_x\frac{\partial u'_x}{\partial x}} + \overline{u'^2_x}\frac{\partial \overline{U_x}}{\partial x} + \overline{u'^2_x\frac{\partial u'_x}{\partial x}} + \overline{U_y}\overline{u'_x\frac{\partial u'_x}{\partial y}} + \overline{u'_x u'_y}\frac{\partial \overline{U_x}}{\partial y} + \overline{u'_x u'_y\frac{\partial u'_x}{\partial y}} \\
&+\overline{U_z}\overline{u'_x\frac{\partial u'_x}{\partial z}} + \overline{u'_x u'_z}\frac{\partial \overline{U_x}}{\partial z} + \overline{u'_x u'_z\frac{\partial u'_x}{\partial z}} = -\frac{1}{\rho}\overline{u'_x\frac{\partial p'}{\partial x}} + \nu\overline{u'_x\left[\frac{\partial^2 u'_x}{\partial x^2} + \frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 u'_x}{\partial z^2}\right]}
\end{aligned} \tag{11}
$$

Using the property:

$$
u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial u^2}{\partial x} \tag{12}
$$

The equation 11 can be further re-arranged

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial t} + \overline{U_x}\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial x} + \overline{U_y}\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial y} + \overline{U_z}\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial z} + \overline{u'^2_x}\frac{\partial \overline{U_x}}{\partial x} + \overline{u'_x u'_y}\frac{\partial \overline{U_x}}{\partial y} + \overline{u'_x u'_z}\frac{\partial \overline{U_x}}{\partial z} \\
&+\left(\overline{u'^2_x\frac{\partial u'_x}{\partial x}} + \overline{u'_x u'_y\frac{\partial u'_x}{\partial y}} + \overline{u'_x u'_z\frac{\partial u'_x}{\partial z}}\right) = -\frac{1}{\rho}\overline{u'_x\frac{\partial p'}{\partial x}} + \nu\overline{u'_x\left[\frac{\partial^2 u'_x}{\partial x^2} + \frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 u'_x}{\partial z^2}\right]}
\end{aligned} \tag{13}
$$

In order to further simplify equation (13), last term of L.H.S of equation (13) can be expanded to:

$$
\begin{aligned}
\left(\overline{u'^2_x\frac{\partial u'_x}{\partial x}} + \overline{u'_x u'_y\frac{\partial u'_x}{\partial y}} + \overline{u'_x u'_z\frac{\partial u'_x}{\partial z}}\right) = &\frac{1}{2}\left(\frac{\partial \overline{u'^3_x}}{\partial x} + \frac{\partial \overline{u'^2_x u'_y}}{\partial y} + \frac{\partial \overline{u'^2_x u'_z}}{\partial z}\right) \\
&-\frac{1}{2}\overline{u'^2_x\left(\frac{\partial u'_x}{\partial x} + \frac{\partial u'_y}{\partial y} + \frac{\partial u'_z}{\partial z}\right)}
\end{aligned} \tag{14}
$$

Again using Reynolds averaging $\overline{\overline{\phi} + \overline{\psi}} = \overline{\phi} + \overline{\psi}$, the second term of R.H.S of above equation (14) becomes:

$$
\begin{aligned}
\left(\overline{u'^2_x\frac{\partial u'_x}{\partial x}} + \overline{u'_x u'_y\frac{\partial u'_x}{\partial y}} + \overline{u'_x u'_z\frac{\partial u'_x}{\partial z}}\right) = &\frac{1}{2}\left(\frac{\partial \overline{u'^3_x}}{\partial x} + \frac{\partial \overline{u'^2_x u'_y}}{\partial y} + \frac{\partial \overline{u'^2_x u'_z}}{\partial z}\right) \\
&-\frac{1}{2}\overline{u'^2_x}\overline{\left(\frac{\partial u'_x}{\partial x} + \frac{\partial u'_y}{\partial y} + \frac{\partial u'_z}{\partial z}\right)}
\end{aligned} \tag{15}
$$

From the divergence rule $\nabla\cdot(U) = \nabla\cdot(\overline{U})$,the second term on the R.H.S of the equation (15) which represents fluctuating components of continuity equation becomes zero.

$$
\begin{aligned}
\left(\overline{u'^2_x\frac{\partial u'_x}{\partial x}} + \overline{u'_x u'_y\frac{\partial u'_x}{\partial y}} + \overline{u'_x u'_z\frac{\partial u'_x}{\partial z}}\right) = &\frac{1}{2}\left(\frac{\partial \overline{u'^3_x}}{\partial x} + \frac{\partial \overline{u'^2_x u'_y}}{\partial y} + \frac{\partial \overline{u'^2_x u'_z}}{\partial z}\right) \\
&-\frac{1}{2}\overline{u'^2_x}\underbrace{\overline{\left(\frac{\partial u'_x}{\partial x} + \frac{\partial u'_y}{\partial y} + \frac{\partial u'_z}{\partial z}\right)}}_{=0}
\end{aligned} \tag{16}
$$

Hence we obtain:

$$
\left(\overline{u'^2_x\frac{\partial u'_x}{\partial x}} + \overline{u'_x u'_y\frac{\partial u'_x}{\partial x}} + \overline{u'_x u'_z\frac{\partial u'_x}{\partial x}}\right) = \frac{1}{2}\left(\frac{\partial \overline{u'^3_x}}{\partial x} + \frac{\partial \overline{u'^2_x u'_y}}{\partial y} + \frac{\partial \overline{u'^2_x u'_z}}{\partial z}\right) \tag{17}
$$

Hence, equation (13) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial t} + \overline{U_x}\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial x} + \overline{U_y}\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial y} + \overline{U_z}\frac{1}{2}\frac{\partial \overline{u'^2_x}}{\partial z} = -\overline{u'^2_x}\frac{\partial \overline{U_x}}{\partial x} - \overline{u'_x u'_y}\frac{\partial \overline{U_x}}{\partial y} \\
&-\overline{u'_x u'_z}\frac{\partial \overline{U_x}}{\partial z} - \frac{1}{2}\left(\frac{\partial \overline{u'^3_x}}{\partial x} + \frac{\partial \overline{u'^2_x u'_y}}{\partial y} + \frac{\partial \overline{u'^2_x u'_z}}{\partial z}\right) - \frac{1}{\rho}\overline{u'_x\frac{\partial p'}{\partial x}} + \nu\overline{u'_x\left[\frac{\partial^2 u'_x}{\partial x^2} + \frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 u'_x}{\partial z^2}\right]}
\end{aligned} \tag{18}
$$

Using the property

$$
\frac{\partial^2(\frac{1}{2}u_i u_j)}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_i}{\partial x_j}\right) = u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \tag{19}
$$

or

$$u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial^2 (\frac{1}{2} u_i u_j)}{\partial x_j \partial x_j} - \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \tag{20}$$

Using equation (20) and further expanding viscous term of the equation (18), we obtain:

$$\begin{aligned}\overline{\nu u_x' \left[\frac{\partial^2 u_x'}{\partial x^2} + \frac{\partial^2 u_x'}{\partial y^2} + \frac{\partial^2 u_x'}{\partial z^2}\right]} &= \nu \left(\frac{\partial^2 \overline{\frac{1}{2} u_x'^2}}{\partial x^2} + \frac{\partial^2 \overline{\frac{1}{2} u_x'^2}}{\partial y^2} + \frac{\partial^2 \overline{\frac{1}{2} u_x'^2}}{\partial z^2} \right) \\ &- \nu \left(\left(\frac{\partial u_x'}{\partial x}\right)^2 + \left(\frac{\partial u_x'}{\partial y}\right)^2 + \left(\frac{\partial u_x'}{\partial z}\right)^2 \right)\end{aligned} \tag{21}$$

Now replacing viscous term in equation (18) by right hand side of equation (21):

$$\begin{aligned}\frac{1}{2}\frac{\partial \overline{u_x'^2}}{\partial t} + \overline{U_x}\frac{1}{2}\frac{\partial \overline{u_x'^2}}{\partial x} + \overline{U_y}\frac{1}{2}\frac{\partial \overline{u_x'^2}}{\partial y} + \overline{U_z}\frac{1}{2}\frac{\partial \overline{u_x'^2}}{\partial z} &= -\overline{u_x'^2}\frac{\partial \overline{U_x}}{\partial x} - \overline{u_x' u_y'}\frac{\partial \overline{U_x}}{\partial y} \\ -\overline{u_x' u_z'}\frac{\partial \overline{U_x}}{\partial z} &- \frac{1}{2}\left(\frac{\partial \overline{u_x'^3}}{\partial x} + \frac{\partial \overline{u_x'^2 u_y'}}{\partial y} + \frac{\partial \overline{u_x'^2 u_z'}}{\partial z}\right) - \frac{1}{\rho}\overline{u_x' \frac{\partial p'}{\partial x}} \\ +\nu \left(\frac{\partial^2 \overline{\frac{1}{2} u_x'^2}}{\partial x^2} + \frac{\partial^2 \overline{\frac{1}{2} u_x'^2}}{\partial y^2} + \frac{\partial^2 \overline{\frac{1}{2} u_x'^2}}{\partial z^2} \right) &- \nu \left(\left(\frac{\partial u_x'}{\partial x}\right)^2 + \left(\frac{\partial u_x'}{\partial y}\right)^2 + \left(\frac{\partial u_x'}{\partial z}\right)^2 \right)\end{aligned} \tag{22}$$

By multiplying u_y' with equation (3) and performing time averaging, the following equation can be obtained:

$$\begin{aligned}\frac{1}{2}\frac{\partial \overline{u_y'^2}}{\partial t} + \overline{U_x}\frac{1}{2}\frac{\partial \overline{u_y'^2}}{\partial x} + \overline{U_y}\frac{1}{2}\frac{\partial \overline{u_y'^2}}{\partial y} + \overline{U_z}\frac{1}{2}\frac{\partial \overline{u_y'^2}}{\partial z} &= -\overline{u_x' u_y'}\frac{\partial \overline{U_x}}{\partial x} - \overline{u_y'^2}\frac{\partial \overline{U_x}}{\partial y} \\ -\overline{u_y' u_z'}\frac{\partial \overline{U_x}}{\partial z} &- \frac{1}{2}\left(\frac{\partial \overline{u_x' u_y'^2}}{\partial x} + \frac{\partial \overline{u_y'^3}}{\partial y} + \frac{\partial \overline{u_y'^2 u_z'}}{\partial z}\right) - \frac{1}{\rho}\overline{u_y' \frac{\partial p'}{\partial y}} \\ +\nu \left(\frac{\partial^2 \overline{\frac{1}{2} u_y'^2}}{\partial x^2} + \frac{\partial^2 \overline{\frac{1}{2} u_y'^2}}{\partial y^2} + \frac{\partial^2 \overline{\frac{1}{2} u_y'^2}}{\partial z^2} \right) &- \nu \left(\left(\frac{\partial u_y'}{\partial x}\right)^2 + \left(\frac{\partial u_y'}{\partial y}\right)^2 + \left(\frac{\partial u_y'}{\partial z}\right)^2 \right)\end{aligned} \tag{23}$$

Similarly the following equation can be obtained by multiplying u_z' by with equation (4) and performing time averaging:

$$\begin{aligned}\frac{1}{2}\frac{\partial \overline{u_z'^2}}{\partial t} + \overline{U_x}\frac{1}{2}\frac{\partial \overline{u_z'^2}}{\partial x} + \overline{U_y}\frac{1}{2}\frac{\partial \overline{u_z'^2}}{\partial y} + \overline{U_z}\frac{1}{2}\frac{\partial \overline{u_z'^2}}{\partial z} &= -\overline{u_x' u_z'}\frac{\partial \overline{U_x}}{\partial x} - \overline{u_y' u_z'}\frac{\partial \overline{U_x}}{\partial y} \\ -\overline{u_z'^2}\frac{\partial \overline{U_x}}{\partial z} &- \frac{1}{2}\left(\frac{\partial \overline{u_x' u_z'^2}}{\partial x} + \frac{\partial \overline{u_y' u_z'^2}}{\partial y} + \frac{\partial \overline{u_z'^3}}{\partial z}\right) - \frac{1}{\rho}\overline{u_z' \frac{\partial p'}{\partial z}} \\ +\nu \left(\frac{\partial^2 \overline{\frac{1}{2} u_z'^2}}{\partial x^2} + \frac{\partial^2 \overline{\frac{1}{2} u_z'^2}}{\partial y^2} + \frac{\partial^2 \overline{\frac{1}{2} u_z'^2}}{\partial z^2} \right) &- \nu \left(\left(\frac{\partial u_z'}{\partial x}\right)^2 + \left(\frac{\partial u_z'}{\partial y}\right)^2 + \left(\frac{\partial u_z'}{\partial z}\right)^2 \right)\end{aligned} \tag{24}$$

Adding equations (18), (23) and (24) and defining turbulent kinetic energy with velocity fluctuations:

$$k = \frac{1}{2}(\overline{u_x'^2} + \overline{u_y'^2} + \overline{u_z'^2}) \tag{25}$$

We obtain:

$$\underbrace{\frac{\partial k}{\partial t}}_{\substack{\text{rate of} \\ \text{change of k}}} \quad \underbrace{+ \overline{U_x}\frac{\partial k}{\partial x} + \overline{U_y}\frac{\partial k}{\partial y} + \overline{U_z}\frac{\partial k}{\partial z}}_{\substack{\text{transport of} \\ \text{k by convection}}} =$$

$$-\underbrace{\left(\overline{u_x'^2}\frac{\partial \overline{U_x}}{\partial x}+\overline{u_x'u_y'}\frac{\partial \overline{U_x}}{\partial y}-\overline{u_x'u_z'}\frac{\partial \overline{U_x}}{\partial z}+\overline{u_x'u_y'}\frac{\partial \overline{U_y}}{\partial x}+\overline{u_y'^2}\frac{\partial \overline{U_y}}{\partial y}+\overline{u_y'u_z'}\frac{\partial \overline{U_y}}{\partial z}+\overline{u_x'u_z'}\frac{\partial \overline{U_z}}{\partial x}+\overline{u_y'u_z'}\frac{\partial \overline{U_z}}{\partial y}+\overline{u_x'^2}\frac{\partial \overline{U_z}}{\partial z}\right)}_{\text{production of k by interaction of Reynolds stress and mean flow}}$$

$$\underbrace{-\frac{1}{\rho}\left[\overline{u_x'\frac{\partial p'}{\partial x}}+\overline{u_y'\frac{\partial p'}{\partial y}}+\overline{u_z'\frac{\partial p'}{\partial z}}\right]}_{\text{transport due to pressure}}$$

$$\underbrace{-\frac{1}{2}\left(\frac{\partial\overline{u_x'^3}}{\partial x}+\frac{\partial\overline{u_x'^2u_y'}}{\partial y}+\frac{\partial\overline{u_x'^2u_z'}}{\partial z}+\frac{\partial\overline{u_x'u_y'^2}}{\partial x}+\frac{\partial\overline{u_y'^3}}{\partial y}+\frac{\partial\overline{u_y'^2u_z'}}{\partial z}+\frac{\partial\overline{u_x'u_z'^2}}{\partial x}+\frac{\partial\overline{u_y'u_z'^2}}{\partial y}+\frac{\partial\overline{u_z'^3}}{\partial z}\right)}_{\text{transport due to Reynolds stress}}$$

$$+\nu\underbrace{\left(\frac{\partial^2\overline{\frac{1}{2}u_x'^2}}{\partial x^2}+\frac{\partial^2\overline{\frac{1}{2}u_x'^2}}{\partial y^2}+\frac{\partial^2\overline{\frac{1}{2}u_x'^2}}{\partial z^2}+\frac{\partial^2\overline{\frac{1}{2}u_y'^2}}{\partial x^2}+\frac{\partial^2\overline{\frac{1}{2}u_y'^2}}{\partial y^2}+\frac{\partial^2\overline{\frac{1}{2}u_y'^2}}{\partial z^2}+\frac{\partial^2\overline{\frac{1}{2}u_z'^2}}{\partial x^2}+\frac{\partial^2\overline{\frac{1}{2}u_z'^2}}{\partial y^2}+\frac{\partial^2\overline{\frac{1}{2}u_z'^2}}{\partial z^2}\right)}_{\text{transport due to viscous stress}}$$

$$\underbrace{-\nu\left[\left(\frac{\overline{\partial u_x'}}{\partial x}\right)^2+\left(\frac{\overline{\partial u_x'}}{\partial y}\right)^2+\left(\frac{\overline{\partial u_x'}}{\partial z}\right)^2+\left(\frac{\overline{\partial u_y'}}{\partial x}\right)^2+\left(\frac{\overline{\partial u_y'}}{\partial y}\right)^2+\left(\frac{\overline{\partial u_y'}}{\partial z}\right)^2+\left(\frac{\overline{\partial u_z'}}{\partial x}\right)^2+\left(\frac{\overline{\partial u_z'}}{\partial y}\right)^2+\left(\frac{\overline{\partial u_z'}}{\partial z}\right)^2\right]}_{\substack{\text{rate of disipation}\\ \text{of k}\\ (\epsilon)}} \tag{26}$$

Equation (26) can also be represented in vector form:

$$\underbrace{\frac{\partial k}{\partial t}}_{\substack{\text{rate of}\\ \text{change of k}}} + \underbrace{\overline{U}\ \nabla\cdot(k)}_{\substack{\text{transport of}\\ \text{k by convection}}} = -\underbrace{\overline{u_i'u_j'}\cdot\overline{E_{ij}}}_{\substack{\text{production of k}\\ \text{by interacton of}\\ \text{Reynolds Stress}\\ \text{and mean flow}}} - \underbrace{\frac{1}{\rho}\nabla\cdot\overline{u'p'}}_{\substack{\text{transport of}\\ \text{k by pressure}}} - \underbrace{\frac{1}{2}\nabla\cdot(\overline{u_i'\cdot u_i'u_j'})}_{\substack{\text{transport of k}\\ \text{by Reynolds Stress}}} + \underbrace{\nu\ \nabla^2 k}_{\substack{\text{transport of k}\\ \text{by viscous stress}}} - \underbrace{2\nu\overline{e_{ij}'\cdot e_{ij}'}}_{\substack{\text{rate of disipation}\\ \text{of k}\\ (\epsilon)}} \tag{27}$$

References

H. Tennekes and J. L. Lumley. *A first course in turbulence*. The MIT press, 1972.